MODERN ANALYTICAL METHODS IN HEAT TRANSFER

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A review is made of certain analytical methods used in heat transfer. Singular perturbation methods, asymptotic methods, and methods associated with the solution of integral equations are considered for the purpose of solving problems in various types of heat transfer.

In the excellent monographs [1-7], the achievements in the various fields of heat transfer are considered from the physical point of view in a systematic and unique way. An attempt is made in this present review to generalize these achievements from the point of view of mathematical methods. It should be emphasized that some of the most important methods in the current literature on heat transfer are still in the process of development and examples from the literature are quoted frequently in order to show the virtue of the method in those branches of heat transfer where it can be considered as a potential means of obtaining a solution. Therefore, in considering a specific method, it is desirable to cite examples from the current literature based on physical considerations. We also try to cite, wherever possible, references to fundamental works connected with the development of methods, for which the author does not make any claims on the completeness of the review in view of the enormous volume of existing literature. The criteria for selecting examples is to show how a specific method has been applied in problems associated with the various types of heat transfer. However, in view of their being generally well known, there will be no need to refer in this review to integral methods, to which the review [8] is devoted, or to a description of their application to nonsteady-state heat transfer [9]. In a similar way, there is no need to refer to classical methods of operational calculus, which is used widely in classical works [10-12]. We shall consider in some detail perturbation methods, asymptotic methods, and methods associated with the solution of integral equations.

1. Perturbation Methods

Perturbation methods consist mainly of the series expansion of dependent variables with respect to powers of known value, assumed to be small. When this small quantity is a parameter, the method is known as "parametric perturbation" and if it is a coordinate the method is called "coordinate perturbation." Assuming that this small quantity is equal to ε, the solution of the differential equation for $\varepsilon \to 0$ is a solution of zero order. When the expansion is inserted in a differential equation and identical powers of ε are equated, we obtain a system of differential equations for solutions of successive orders. The series obtained is convergent in the asymptotic sense [96] and if the scheme mentioned above is suitable, then it is called a regular perturbation. This method was used in a number of problems and has led to very useful results.

However, in many problems the ratio of successive terms ceases to be small and the system of regular perturbation becomes untenable in a certain region of the flow field. Thus, it is impossible to obtain a valid solution over the whole flow field by the method of regular perturbation. These problems are known as problems of singular perturbation.

Sometimes the situation described above arises because of the presence of singularity in a solution of zero order at a point or on a line in the region of investigation. This singularity becomes greater according as the order of the solution increases. A procedure for solving such problems by the perturbation method was proposed in [13], according to which a dependent variable $v(x, \varepsilon)$ and an independent variable x are expanded in powers of low value in a series of the form:

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$$v = v_{(0)}(\chi) + \varepsilon v_{(1)}(\chi) + \varepsilon^2 v_{(2)}(\chi) + \ldots,$$
$$x = \chi + \varepsilon x_{(1)}(\chi) + \varepsilon^2 x_{(2)}(\chi) + \ldots,$$

where χ replaces the initially independent variables x, and $v_{(0)}(\chi)$ is simply a solution of zero order of the classical method of perturbation with χ instead of x. The method is described in [14] with various applications, where the author calls it the Poincairé–Lighthill–Kuo method, the the PLK method, because Kuo applied it to supersonic boundary layers. The method is frequently called the method of deformed coordinates (see, for example, Chapter VI of [17] where many examples are given of the application of this method in aerodynamics). Lighthill [18] applied this method to conical shock waves in a steady-state supersonic flow. In [19, 20] it is applied to a supersonic streamline flow around a profile. The limitation of this method is shown by Lighthill [21] himself, who finally recommended the use of the method only in hyperbolic equations. This method is used in [22, 23] for formulating the correctness of the solution to the problem of temperature distribution in floating bodies using the integral method by Morris [24] to obtain a uniformly convergent solution of laminar convective flow in a heated vertical tube rotating round a parallel axis.

Olstad [25] considered the problem of radiative flow at the point of interruption as a perturbation occurring without emission. He found that near the wall, the method of regular perturbation is inapplicable and he used the PLK method for obtaining a homogeneous solution.

If the leading derivative of the differential equation contains a small parameter, then the PLK method is inapplicable. The main difficulty in these problems appears because when $\varepsilon \to 0$ the order of the equation is reduced and in this way certain boundary conditions may not be satisfied. A method was developed in [26-29], known under the name of "method of combined asymptotic expansions."

Suppose $v(x, \varepsilon)$ is the solution of the problem of singular perturbation. The general asymptotic expansion in powers of ε when $\varepsilon \to 0$ is called an external expansion for a fixed value of $x > 0$. This expansion is valid over the range $\gamma \le x \le 1$ with γ independent of ε. The expansion can also be fulfilled for $\gamma \le x \le 1$, even if γ depends on ε and approaches zero, when $\varepsilon \to 0$ for the condition $\gamma/\varepsilon \to \infty$ (Erdelyi [30]). Suppose the external expansion be denoted by v^0. In order to obtain the internal expansion, the expanding transformation $x = z\varepsilon$ is introduced. The asumptotic expansion $v(z\varepsilon, \varepsilon)$ for $\varepsilon \to 0$ when $z \ge 0$ is fixed, is called the internal expansion and is denoted by v^i. This expansion is valid for $0 \le (z = x/\varepsilon) \le A_1$. It was shown by examples that the expansion is fulfilled in the region $x/\varepsilon = O(1)$. Thus, the internal and external expansions have a common region of applicability and in this region we can write the internal expansion of the external expansion v^0, i.e., $(v^0)^i$, and the external expansion of the internal expansion, i.e., $(v^i)^0$. The asymptotic principle of combination [17] establishes that

the m-th term of the internal expansion (n-th term of the external expansion) = n-th term of the external expansion (m-th term of the internal expansion). (3)

Here, m and n are any two whole numbers. In practice, m is usually chosen as n or as n + 1. The unknown constants in v^0 and v^i are determined by the congruence of this pair in the common region. Sometimes, combined expansions v^c are formed in order to obtain a solution which is homogeneous over the whole range $0 \le x \le 1$. v^c can be formed either according to the law of additivity

$$v^c = v^0 + v^i - (v^0)^i, \tag{4a}$$

or according to the law of multiplicativity

$$v^c = v^0 v^i / (v^0)^i, \tag{4b}$$

as discussed in detail in [17].

Lam [31] considered the internal and external expansion of the boundary layer at the walls of supersonic nozzles in the case of a special relation between the interaction of heat transfer and the boundary layers with very favorable pressure gradients. This method was used in [32] to obtain a uniformly converging solution of laminar flow in a homogeneous porous channel with a large (air) injection. The method of extension of internal coordinates is used in [33] in order to obtain the internal solution of the problem of natural convection over a flame. Mueller and Malmuth [34] considered the temperature distribution in a radiating heat shield with a random aerodynamic source and longitudinal thermal conductivity. Whilst the problem for small radiation reduces to the problem of a regular perturbation, the problem for low conductivity is a problem of singular perturbation for which the method of combined asymptotic expansions was used. It is shown in [35] that the solution of the dimensionless equation of a boundary layer describing the free convection of a radiating (sulfur) liquid is itself a problem of singular perturbation. Burgraff [36] considered viscous flow of a transparent layer as an approximated model of a shock layer. In the case of constant density, he obtained a precise solution and explained the interaction between the viscous and non-viscous regions by considering the asymptotic expansion of the above-mentioned solution with respect to values of the Reynolds number. Then he converted to the construction of a solution of the general case by means of the method of combined asymptotic expansions. Inger [37] applied this method to the analysis of dissociating boundary layers close to equilibrium. We shall discuss this paper is some detail in order to illustrate the method.

Let us consider the flow in a boundary layer close to equilibrium of a dissociating diatomic gas along an impermeable axisymmetrical or two-dimensional body which is either adiabatic or has a uniform surface temperature.

If we introduce the variables

$$\eta = \rho_e u_e r_B^j (2\xi)^{-1/2} \int\limits_0^y (\rho/\rho_e)\, dy, \tag{5}$$

$$\xi = C \int\limits_0^x \rho_e u_e \mu_e r_B^{2j}\, dx, \tag{6}$$

$$u = u_e \frac{df}{d\eta} = u_e f' \tag{7}$$

and assume that Pr = 1, Le = 1, and $\rho\mu$ = const, we can write the equations of momentum, inertia, and atomic concentration in the form

$$ff' + f''' = 0, \tag{8}$$

$$f\frac{\partial \alpha}{\partial \eta} + \frac{\partial^2 \alpha}{\partial \eta^2} - 2\xi f' \frac{\partial \alpha}{\partial \xi} = \overline{\Gamma}\, \xi^R (\alpha - C_1 - C_2 T), \tag{9}$$

$$f\frac{\partial H}{\partial \eta} + \frac{\partial^2 H}{\partial \eta^2} - 2\xi f' \frac{\partial H}{\partial \xi} = 0. \tag{10}$$

The total enthalpy H is related with the static temperature and atomic mass fraction α by the relation

$$H = c_p T + \alpha h_D + \frac{1}{2} u_e^2 (f')^2. \tag{11}$$

It should be noted that $\overline{\Gamma} \to 0$ for a chemically frozen flow and $\overline{\Gamma} \to \infty$ for total equilibrium. The boundary conditions have the form

$$\begin{gathered} f'(\infty) = 1,\ \alpha(\xi, \infty) = \alpha_e = C_1 + C_2 T_e, \\ T(\xi, \infty) = T_e, \\ H(\xi, \infty) = H_e = c_p T_e + \alpha_e R_D + \frac{1}{2} u_e^2. \end{gathered} \tag{12}$$

At the surface

$$f(0) = f'(0) = 0, \tag{13}$$

$$\begin{gathered} T(\xi, \infty) = T_w = \text{const} \quad \text{or} \quad \frac{\partial H(\xi, 0)}{\partial \eta} = 0, \\ H(\xi, 0) = c_p T_w + h_D \alpha(\xi, 0). \end{gathered} \tag{14}$$

For a completely catalytic wall we also have

$$\alpha(\xi, 0) = \alpha_{EQ},\ w = C_1 + C_2 T_w. \tag{15}$$

Thus, in Eq. (9) for $\overline{\Gamma} \to \infty$ ($1/\overline{\Gamma} \to 0$) the leading derivative vanishes and therefore the problem is a problem of singular perturbation. We can write Eq. (9) and (10) by means of the new independent variables $\bar{\alpha}$, G, and Γ

$$f\frac{\partial \bar{\alpha}}{\partial \eta} + \frac{\partial^2 \bar{\alpha}}{\partial \eta^2} + 2\xi f' \frac{\partial \bar{\alpha}}{\partial \xi} = \Gamma \xi^R \left(\bar{\alpha} - \frac{DG}{1+D}\right) - (f\alpha'_{EQ} + \alpha''_{EQ}), \tag{16}$$

$$f\frac{\partial G}{\partial \eta}+\frac{\partial^2 G}{\partial \eta^2}-2\xi f'\frac{\partial G}{\partial \xi}=0 \tag{17}$$

with the boundary conditions

$$\bar{\alpha}(\xi,\ \infty)=0, \tag{18}$$

$$\bar{\alpha}(\xi,\ 0)=0 \text{ (for a catalytic wall)}, \tag{19}$$

noting that the temperature profile can be represented as

$$T(\xi,\ \eta)=T_{EQ}(\eta)+c_p^{-1}(Gh_D-\bar{\alpha}h_D). \tag{20}$$

It follows from Eq. (17), therefore, that

$$G(\xi,\ \eta)=0. \tag{21}$$

We consider Eq. (16) for a flow approaching equilibrium, where Γ is very large. An internal expansion can be assumed, therefore, for

$$\bar{\alpha}^0-D(1+D)^{-1}G=\sum_{N=1}^{\infty}\bar{\alpha}^0_{(N)}(\eta)(\Gamma\xi^R)^{-N}. \tag{22}$$

Substituting Eq. (22) in Eq. (16), using Eq. (17) and collecting terms with identical powers, we find the following equations defining the perturbation functions

$$\bar{\alpha}^0_{(1)}(\eta)=f\alpha'_{EQ}+\alpha''_{EQ}=\alpha''_{EQ}(0)[f''(\eta)/A]^2, \tag{23a}$$

$$\bar{\alpha}^0_{(2)}(\eta)=f(\bar{\alpha}^0_{(1)})'+(\bar{\alpha}^0_{(1)})''+2Rf'\bar{\alpha}^0_{(1)}, \tag{23b}$$

$$\bar{\alpha}^0_{(N)}(\eta)=f(\bar{\alpha}^0_{(N-1)})'+(\bar{\alpha}^0_{(N-1)})''+2R(N-1)\bar{\alpha}^0_{(N-1)}, \tag{23c}$$

where Eq. (20) was used to simplify the right-hand side of Eq. (23a). Although Eq. (22) satisfies the external boundary condition (18), it does not satisfy the internal boundary condition (19). Therefore, we shall attempt to obtain a solution in the vicinity of the wall in terms of the compression by means of the variable Q. It follows from Eq. (16) that the corresponding compression can be obtained if we put $Q = (\Gamma^{1/2}\eta)$.

In order to obtain the internal solution in terms of the new independent variable, we first of all rewrite Eq. (16) in terms of the new independent variable $Q = \Gamma^{1/2}\eta$. Thus, we obtain the equation

$$\frac{\partial^2\alpha}{\partial Q^2}+\frac{AQ^2}{2\Gamma^{3/2}}\left(1-\frac{2BQ^3}{A\Gamma^3}\right)\frac{\partial\bar{\alpha}}{\partial Q}-\frac{2\xi AQ}{\Gamma^{3/2}}\left(1-\frac{5BQ^3}{A\Gamma^{3/2}}\right)\frac{\partial\bar{\alpha}}{\partial\xi}$$
$$=\xi^R\left[\bar{\alpha}-\frac{DG(\xi,\ 0)}{1+D}-\frac{DQ\Gamma^{-1/2}}{1+D}\frac{\partial G(\xi,\ 0)}{\partial\eta}-\frac{(\bar{\alpha}^0_{EQ})''}{\Gamma}\right], \tag{24}$$

which is satisfied by the boundary conditions at the wall (19). Equation (24) is now solved for condition (19). If we assume that

$$\alpha=\sum_{N=1}^{\infty}\bar{\alpha}^i_{(N)}(\xi,\ Q)\Gamma^{-\frac{1}{2}N}, \tag{25}$$

then substitution of the series (25) in Eq. (24) gives a sequence of linear second-order differential equations, defining the internal perturbation functions. Solving these equations, we obtain

$$\bar{\alpha}^i_{(1)}(\xi,\ Q)=E_1\sinh(\xi^{\frac{1}{2}R}Q), \tag{26a}$$

$$\bar{\alpha}^i_{(2)}(\xi,\ Q)=E_2\sinh(\xi^{\frac{1}{2}R}Q)+\xi^{-R}\alpha''_{EQ}(0)[1-\exp(-\xi^{\frac{1}{2}R}Q)], \tag{26b}$$

$$\alpha^i_{(3)}(\xi,\ Q)=E_3\sinh(\xi^{\frac{1}{2}R}Q), \tag{26c}$$

where E_1, E_2, ... are arbitrary constants. Now we match this internal solution with the external solution by means of the principle of asymptotic combination which is given by the rule (3).

From Eq. (21), (22), and (23) we have the external solution for the atomic concentration

$$\bar{\alpha}^0(\xi,\ \eta)=[\alpha''_{EQ}(0)/(\Gamma\xi^R)][f''(0)/A]^2+O(\Gamma^{-2}\xi^{-2R}), \tag{27}$$

which, rewritten in terms of the internal variable Q and expanded for large values of Γ using the fact that $f'''(0) = 0$, assumes the form

$$\bar{\alpha}^i(Q, \xi) = \alpha''_{EQ}(0)(\Gamma\xi^R)^{-1} + O(\Gamma^{-2}\xi^{-2R}). \tag{28}$$

From Eqs. (25), (26a), (26b), and (26c), the corresponding internal solution is

$$\bar{\alpha}^i(Q, \xi) = E_1\Gamma^{-1/2}\sinh(\xi^{\frac{1}{2}R}Q) + E_2\Gamma^{-1}\sinh(\xi^{\frac{1}{2}R}Q)$$
$$+\Gamma^{-1}\xi^{-R}\alpha''_{EQ}(0)\left[1-\exp(-\xi^{\frac{1}{2}R}Q)\right] + E_3\Gamma^{-3/2}\sinh(\xi^{\frac{1}{2}R}Q) + O(\Gamma^{-2}), \tag{29}$$

which, in terms of the internal variable for large values of Γ is expressed as

$$\bar{\alpha}^i(\eta, \xi) = \begin{cases} \sinh(\eta\xi^{\frac{1}{2}R}\Gamma^{1/2})\sum_N E_N\Gamma^{-\frac{1}{2}N}, & E_N \neq 0; \\ (\Gamma\xi^R)^{-1}\alpha''_{EQ}(0)\left[1-\exp(-\xi^{\frac{1}{2}R}Q)\right], & E_N = 0. \end{cases} \tag{30}$$

Neglecting exponentially small terms, Eq. (28) and (30) will be combined if $E_1 = E_2 = E_N = 0$. The final internal solution for a catalytic wall is:

$$\bar{\alpha}^i_{(1)}(Q, \xi) = 0, \tag{31a}$$

$$\bar{\alpha}^i_{(2)}(Q, \xi) = \alpha''_{EQ}(0)\xi^{-R}\left[1-\exp(-\xi^{\frac{1}{2}R}Q)\right], \tag{31b}$$

$$\bar{\alpha}^i_{(3)}(Q, \xi) = 0. \tag{31c}$$

Knowing that f is the solution of the Blasius equation, we determined the uniformly correct solution comprising Eq. (23) and (31).

Ellinwood [38] used a pair of combined asymptotic expansions in order to obtain solutions for supersonic flows passing through shock layers formed around a blunt, narrow cone or a wedge. In [39] the problem of heat transfer from a heated stationary sphere in a fluctuating flow is studied.

The solution of the boundary-layer equations for large values of the pressure gradient parameter gives one further example of the problem of singular perturbation. This was mentioned by Coles [40] and later by Bekweth and Cohen [41]. For a detailed discussion, see Lagerstrom [42] and Dewey and Gross [43].

The flow in a laminar two-dimensional boundary layer with radiation is analyzed by Novotany and Yang [44]. They assumed small temperature differences inside the flow field. By considering an optically thin approximation they encountered the problem of singular perturbation of the energy equation. This problem was analyzed by combined asymptotic expansions with respect to a parameter defining the optical thickness of the gas.

Fendell [45] used the same method in solving the problem of laminar natural convection around an isothermally heated sphere for a small Grasshof number. The problem of dynamically similar compressed boundary layers with a large injection and with a suitable pressure gradient was considered by Kubota and Fernandez [46], who obtained combined asymptotic expansions for each of two layers: a) the inner layer, adjacent to the surface where viscosity is important and b) the outside boundary layer at which transfer takes place from the inside layer to the outside flow. Starting from the method developed previously in [34], Mueller and Malmuth [47] discussed the approximate solution for thermal conductivity and radiating shells relating to a discrete solar flow. Kueken [48] applied this method to a free-convection boundary layer for the case when the Prandtl number tends to zero.

Vasil'eva [49] justified the method of finding the equilibrium solutions of a system of differential equations containing a small parameter as the leading derivative. Using this method, Varma and Murgai [50] obtained an analytic solution of the problem of natural convection above jets containing solid particles. A detailed list of investigations by Soviet mathematicians and scientists into the mathematical theory of perturbation methods is given in [49].

In the papers mentioned below, some basic theories were discussed of singular perturbation methods, which are of definite interest for application. The following problem is investigated in [51]: we consider the general differential equation of m-th order, depending on the parameter ε in such a way that its order has been reduced to n when ε → 0. Although the starting equation includes m boundary conditions, the derived equation may contain only n boundary conditions. Thus, in the limit $\varepsilon \to 0$, $m - n$ conditions are lost but we wish to determine how each of them is lost. The conditions were obtained which guarantee that the solution $v(x, \xi)$ exists and also is a homogeneous asymptotic expansion.

This method was recently critically examined by Frankel [52], where satisfactory conditions are strictly substantiated for which the van Dyke principle of asymptotic combination [17] is valid. He applied this method to a normal differential equation with a reversal point and showed that the bounded principle of combination is valid, even when it is applied to trimmed internal and external expansions which do not span the order of the terms necessary for combination.

The method of multiple scales is developed by Cochran [53] and Mahony [54]. In their solutions the sensitive coordinate was taken up by a pair of coordinates, thus increasing the number of independent variables. It was then assumed that the general asymptotic expansion is homogeneous over the entire region. Consequently, the need for combination was removed. A similar idea was put forward in [55]. These considerations were developed in [56]. A similar method, called the method of intermediate limits, was described by Kaplun [29, 57]. Other important references are [58-68].

Perturbation methods are only partially applicable in elliptical flow problems.

In the case of problems in which the expansion of a power series is represented in a differential equation, it is found that an equation of n-th order contains terms of (n + 1)-th order. This case is contrary to the case of parabolic equations, where equations of n-th order contain terms of up to the n-th order, as in the case of the Blasius expansion in boundary-layer theory. It is important, therefore, somehow to terminate the series at terms of defined order, so as to match the number of unknowns with the number of equations. This method is sometimes called the method of series termination [75]. Swigart [69] and Bazzhin and Gladkov [70] first used the method of series termination to consider the reverse problem of two- and three-dimensional, nonviscous, axisymmetrical, nonradiative flow around blunt bodies. In this approximation, the independent variables, stream functions, and density ρ are first expanded in power series with respect to a longitudinal cylindrical coordinate ξ or with respect to trigonometrical functions of an angle. Substitution of this series in the defining differential equations in partial derivatives and in the combination of terms with identical powers in ξ, gives the differential equations with the common coordinate η as the independent variable. Termination of a series at a defined order gives a closed system of equations which is solved numerically. In [71] and [72] this method is applied to viscous flow and nonequilibrium reacting flows respectively. Conti [72] found that if the pressure and not the density is expanded in power series, then the accuracy at each termination increases significantly. Van Dyke [73] was able to achieve a significantly higher accuracy by means of a second-order termination, using the pressure as the main variable and also $\xi^2/(1 + \xi^2)$ as the variable of the expansion instead of ξ and by substituting ψ for η. Cheng and Vincenti [74], having closely adhered to van Dyke's scheme, extended the method to the problem of radiative viscous flows around blunt bodies.

However, in all the above-mentioned cases the series are terminated arbitrarily at a higher order but only in order to reduce the number of variables. In this case, the magnitude of the missing terms is just as great as the terms retained. The author of [75] showed that if the equation of momentum is replaced by Bernouil's equation, the the higher-order terms in each termination become velocity components normal to the body or to the shock wave. As these terms are in reality very small in the case of supersonic flows, it is possible to use order-of-magnitude analysis. It was shown that if other terms of the order of the discarded terms also are missing, then an analytic solution can be obtained in certain cases even up to the third order. The basis of the theory of perturbations of elliptical equations has been discussed recently from the mathematical point of view by Ton [76] and Eckhans and de Jager [77].

2. Methods Associated with the Solution of Integral Equations

The maximum utilization of methods associated with the solution of integral equations is related to the theory of heat transfer with radiation. This is because the equations of transfer for radiative flow, according to their internal nature, are formulated as integral-differential equations which, in certain general cases, reduce to integral equations. Some noteworthy reviews have appeared recently on heat transfer with radiation, [78] and [79], which describe the present-day advances in the field of mathematical methods using these integral equations. A detailed account of these methods can be found also in [80, 81, and 82], where at the same time certain very controversial mathematical methods are considered for obtaining precise solutions of the equations of heat transfer with radiation in special cases.

In cases of radiant exchange between series of surfaces or series of diffusion cavities from isothermal surfaces and a nonuniformly distributed flow, the problem of finding the temperature distribution is reduced to the solution of an integral equation of the Fredholm type. Variation methods of solution in this case are discussed in a review by Sparrow [83] and Howell and Siegel [84]. Methods of solution usually include numerical integration, the use of approximate factorization of kernels, approximate solutions of variation methods and expansion in Taylor series. Detailed information relating to this can be obtained from [86-90]. Carrier [85] considered two approximate methods, namely the method of kernel substitution and the method of integral equations of a boundary layer, which are particularly suitable for integral equations in the case of transfer with radiation.

In view of what has been said above, we shall not attempt to discuss further the problem of radiation in the form that it becomes in the formulation of integral methods. However, we shall present certain examples in which differential equations and partial derivatives were converted to integral equations in order to use existing methods for solving these equations.

Boley [91] transformed the differential equation of thermal conductivity into partial derivatives in the region of melting or solidification into an integral–differential equation, the solution of which is in the form of series. Koh and Hartnett [92] reduced the equations of momentum and energy for laminar flow at the surface of penetration with suction to ordinary integral equations which were solved numerically by the iteration method. The same approach was used by Eckert et al. [93, 94] for a two-component boundary layer at a surface with temperature-dependent physical properties. Tolubinskii [95, 96] gave a very general method of solving the integral equations on the assumption of a finite rate of diffusion. Assuming that the solution of the problem for infinite space is known, the corresponding solution for any region can be constructed. Grinchenkov and Ulitko [97] reduced the problem of an established temperature distribution in a semi-infinite medium when its surface is maintained at a temperature of zero but a disc $0 < r < a$ or a ring $a < r < b$ is maintained at constant temperature, to Fredholm's integral equation of second order. Vasilevskii [98] reduced nonlinear parabolic equations of combined heat and mass transfer under nonsteady-state conditions and with boundary conditions of the first order, to a system of two ordinary integral–differential equations, using Boltzman's similarity transformation and he obtained approximate analytic solutions of these equations. Patankar [99], using a two-parameter profile for temperature, reduced the analysis of heat transfer through turbulent boundary layers with a temperature discontinuity at the wall to integral equations in order to determine the parameters in the profile. These integral equations were solved for a laminar velocity profile, a velocity profile varying according to a "seventh power" law and according to some law at the wall by Spalding [100]. Savino and Siegel [101] introduced integral equations for the temperature distribution in a solidifying layer formed in a hot liquid moving along an isothermal cold surface and compared his results with an earlier variational solution of this problem.

Lighthill [102] suggested a new method based on singular differential equations of the Volterra type for heat transfer in boundary layers at a surface whose temperatures varies along its length. This method subsequently was used in many other applications and therefore can be justly called the Lighthill–Volterra method. Using linear approximation for the velocity near the surface in a laminar boundary layer

$$u + \tau_w(x)\, y/\mu, \tag{32}$$

Lighthill showed that the magnitude of the local heat transfer at the wall is

$$Q_w(x) = -K_f\left(\frac{\sigma\rho}{q\mu^2}\right)^{1/3}\frac{[\tau_w(x)]^{1/2}}{\Gamma(4/3)}\int_0^x\left(\int_{x_1}^x[\tau_w(z)]^{1/2}dz\right)d\Big(T_\varepsilon(x_1)\Big), \tag{33}$$

where $T_\varepsilon(x) = T_W(x) - T_\infty$, and the integral in the equation given above is the Stieltjes integral in the sense

$$\int_0^x f(t,\ x)\, dg(t) = f(0,\ x)\, g(0) + \int_0^x f(t,\ x)\, g'(t)\, dt. \tag{34}$$

Formula (33) for heat transfer is, by its nature, approximate for high Prandtl numbers. With increase of σ the thermal boundary layer becomes thinner by comparison with the velocity boundary layer and therefore the linear approximation for the velocity becomes even more and more precise. Lighthill noted that in the stated formula μ and ρ appear only in the form of a derivative. If this derivative is a constant, then the solution of Eq. (33) for the law of velocities satisfies all Mach numbers. Thus, for a plane plate, where

$$\tau_w(x) = 332\left(\frac{\mu\rho u_\infty^3}{x}\right)^{1/2}, \tag{35}$$

formula (34) is simplified to

$$Q_w(x) = 339 \frac{K_f}{\mu} \sigma^{1/3} (\mu\rho u_\infty)^{1/2} x^{-1/4} \int_0^x (x^{3/4} - x_1^{3/4})^{-1/3} \frac{dT_w(x)}{dx_1} dx_1. \tag{36}$$

The same equation was derived by Rubesin and Inouye [103], by means of separation of variables. Tifford [104] suggested a procedure for applying Lighthill's method to a boundary layer with a pressure gradient by introducing into Eq. (33) a quantity which takes into account the pressure gradient. An important contribution for improving the accuracy of Lighthill's method and its application to a boundary layer with a pressure gradient was made by Spalding [105], Liepmann [106], and Davies and Bourne [107]. This method is extended by Illingworth [108] and Lilley [109] to compressed boundary layers with a pressure gradient and heat transfer. A brief assessment of these methods is given by Curle [110].

Lighthill's integral relations were applied [111-115] to chemically active boundary layers with reactions at the surface. A precise solution of the integral equation was obtained only for a plane plate in [111, 112]. Frank-Kamenetskii [113] obtained the results of numerical integration of the equation for reactions of orders of 0.5 and 2.

Chambre [116] and Mann and Wolf [117] reduced the solution of the problem of heat conductivity in a semi-infinite solid body with nonlinear boundary conditions to a similar type of integral Volterra equation. The reverse problem of heat exchange was reduced in [118, 119] to the solution of a singular integral equation of the Volterra type. A similar integral equation, describing the effect of a catalysis discontinuity at the surface in the flow of a layer of dissociating gas, was solved by the method of expansion in series, and also numerically in [120]. Perel'man [121] showed that the solution of the conjugate problem of heat transfer between the surface and the flow of a boundary layer can be reduced to the solution of these equations. The general theory of conformal equations is considered in detail by Mikhlin [87] and Muskhelishvili [88].

The solution in the form of series in equations of the above-mentioned type has normally a small radius of convergence and is suitable only for small values of the argument. The main difficulty in solving equations of this type for a large or moderately large argument is that the asymptotic form of the solution cannot be substituted directly in the equation, because it is necessary to consider the contribution of the integral near the lower limit. Thus, it is obvious that in these equations the asymptotic solution for $x \to \infty$ depends on the behavior of the solution near the origin of the coordinates. Therefore, these equations normally are solved by numerical methods. Perel'man [121, 122] produced a method based on Mellin's transformation for finding asymptotic solutions in linear singular integral equations of the Volterra type. Kumar [123] used this method in solving the conjugate problem of heat transfer in a laminar boundary layer above a porous plane plate with air injection. It should be pointed out that the Perel'man method is not applicable for obtaining a solution close to the origin of the coordinates and the essential aspect of this method, namely transformation of the functional equation for the Mellin transformant f(S) to another transformant $f_1(S)$, where $f_1(S)$ is regular in $S > 0$, is not uniquely possible.

Recently, Kumar and Bartman [124] used a new method for the asymptotic solution of these equations close to the origin of the coordinates and for large values of x. This method was developed by Kumar [125] and is applied to the solution of various problems in nonlinear thermal conductivity and surface chemical reactions in compressed boundary layers. It is illustrated below in the solution of Lighthill's problem [102] for finding the radiative–convective equilibrium temperature distribution in such a way that the energy emitted from the plate, in accordance with the Stefan–Boltzman law, is balanced by the incoming thermal flow. Using Lighthill's integral relation [102], the problem is reduced to solution of the equation*

$$[F(z)]^4 = -\frac{1}{2z^{1/2}} \int_0^z \frac{F'(z_1)\, dz_1}{[z^{3/2} - z_1^{3/2}]^{1/3}}, \quad F'(z) = \frac{d}{dz}(F), \tag{37}$$

where

$$F(z) = T(z)/T_r. \tag{38}$$

Equation (37) must be solved with the boundary conditions:

$$F(0) = 1, \tag{39}$$

$$F(\infty) \to 0. \tag{40}$$

*We have introduced certain refinements into the discussion of the example (editor).

Using Abel's transformation formula and integrating with the use of Eq. (39), we can write

$$\Phi(z)=1-F(z)=\frac{3\sqrt{3}}{2\pi}\int_0^z \frac{z_1F^4(z_1)\,dz_1}{(z^{3/2}-z_1^{3/2})^{2/3}}. \tag{41}$$

Formally, Mellin's transformation Eq. (41) can be written as in [126]

$$\varphi(S)=\ \mathrm{M}[\Phi]=\frac{\sqrt{3}}{\pi}B\left[\frac{2}{3}-\frac{2}{3}S,\ \frac{1}{3}\right]\mathrm{M}[F^4z^{-1}], \tag{42}$$

where the symbol M[F] denotes Mellin's transformation $\int_0^\infty F(z)z^{S-1}dz$ of the function F(z). If we assume that

$$\Phi(z)\simeq\sum_{n=1}^{\infty}A_nz^{an},\qquad z\to 0; \tag{43}$$

$$F(z)\simeq\sum_{n=1}^{\infty}B_nz^{-bn},\qquad z\to\infty, \tag{44}$$

then, in accordance with the principle of analytic continuation for Mellin's transformation [127, 124, 129], the poles and residues of both parts of Eq. (42) must be expressed by A_n, a; B_n, b. This should give us a system of equations for the set of unknown parameters in Eq. (43) and (44). Now, by performing analytic continuation of the common band of analyticity D of both parts of Eq. (42) (this band is defined as the region of absolute convergence of the defining Mellin integral) and equating in both parts of Eq. (42) the consistently occurring values of the poles and residues, we also obtain the same system. Thus, continuation of D in the left-hand half-plane in matching the form of the first pole in both parts of Eq. (42) determines a = 1, after which comparison of the residues in successive poles S = $-a$, $-2a$, ... leads to concurrent relations for A_n. Thus, we have

$$A_1=\frac{2\Gamma(4/3)}{\Gamma(2/3)\Gamma(5/3)}=1.461,$$

$$A_2=\frac{-2\Gamma(2)}{\Gamma(2/3)\Gamma(3)}4A_1=-7{,}252,$$

$$A_3=\frac{2\Gamma(10/3)}{\Gamma(2/3)\Gamma(3)}(6A_1^2-4A_2)=46.46.$$

So that when z → 0

$$F(z)=1-1.461z+7.252z^2-46.46z^3+\cdots \tag{45}$$

In order to obtain the solution for large values of z we perform the analytic continuation of D to the right. Thus, we find b = 1/4 and

$$B_1^4=1/2,$$

$$4B_1^3B_2=\frac{1}{2}\frac{\Gamma(2/3)\Gamma(5/6)}{\Gamma(1/2)}B_1,$$

$$4B_3B_1^3+6B_2^2B_1^2=\frac{1}{2}\frac{\Gamma(2/3)\Gamma(2/3)}{\Gamma(1/3)}B_2,$$

$$4B_4B_1^3+12B_3B_2B_1^2+4B_2^3B_1=\frac{1}{2}\frac{\Gamma(2/3)\Gamma(1/2)}{\Gamma(1/6)}B_3,$$

giving

$$F(z)=0{,}8409z^{-1/4}-0.15242z^{-1/2}-0.195z^{-3/4}-0.0038z^{-1}+\cdots \tag{46}$$

The solutions of Eqs. (45) and (46) are exactly the same as obtained by Lighthill [102]* by quite complex analysis of a series of integral values. It should be noted that the application of Abel's transformation

*Strictly speaking, in what follows after the extracted terms of the expansion (46), the coefficients B_n become functions of z – polynomials of ln z. This question is discussed in the paper by Perel'man, Bartman, and Levitan [129] and is there compared with the numerical results (editor).

formula to Eq. (37) and integration, in order to include the boundary condition (39) in the new integral equation (41), are important aspects in this present method, which takes account of the dependence of the solution for large values of z on the solution for $z \to 0$. Kumar [125] showed that this method can be applied to a large number of problems and analytical results can be obtained where, until recently, only numerical solutions were available. In addition, Kumar [128] applied this method to a boundary layer of dissociating gas with disruptive catalysis at the wall and obtained analytic solutions, which coincide well with the precise numerical results of [120].

NOTATION

v^0	is the external expansion;
v^i	is the internal expansion;
v_c	is the combined expansion;
ξ	is the dimensionless coordinate along the body (Eq. 6);
η	is the dimensionless coordinate, normal to the body (Eq. 5);
C	is the Chapman–Rubesin constant = $\rho u/\rho_e u_e$;
ρ	is the density;
u	is the velocity along a direction;
f	is the self-similar variable (Eq. 7);
r_B	is the radius of the axisymmetrical body;
j	is the suffix of r_B, j = 0 for two-dimensional flow and j = 1 for axisymmetrical flow;
T	is the temperature;
c_p	is the specific heat at constant pressure;
α	is the atomic concentration;
$\alpha_{EQ} = C_1 + C_2T$	is the equilibrium atomic concentration;
Γ	is the Damkohler number;
w	is the bulk dissociation;
H	is the total enthalpy (Eq. 11);
h_D	is the energy of dissociation per unit atomic mass;

$\Gamma = \bar{\Gamma}(1 + D)$;
$D = h_D(\alpha_e - \alpha_{EQ,w})/c_p(T_e - T_w)$;
$\bar{\alpha} = \alpha(\xi, \eta) - \alpha_{EQ}(\eta)$;
$Q = (\Gamma^{1/2}\eta)$;

χ	is the flow function;
$\tau_w(x) = \mu(\partial u/\partial y)_{y=0}$	is the surface friction at the wall;
μ	is the coefficient of friction;
$Q_w(x)$	is the thermal flux at the wall;
σ	is the Prandtl number;
T_∞	is the temperature of main stream;
K_f	is the thermal conductivity of fluid;
u_∞	is the free flow velocity;
S	is the parameter for Mellin's transform;
Γ	is the gamma function;
B	is the beta function.

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